Purely infinite C^*-algebras of real rank zero

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Abstract

We show that a separable purely infinite C^*-algebra is of real rank zero if and only if its primitive ideal space has a basis consisting of compact-open sets and the natural map $K_0(I) \to K_0(I/J)$ is surjective for all closed two-sided ideals $J \subset I$ in the C^*-algebra. It follows in particular that if A is any separable C^*-algebra, then $A \otimes \mathcal{O}_2$ is of real rank zero if and only if the primitive ideal space of A has a basis of compact-open sets, which again happens if and only if $A \otimes \mathcal{O}_2$ has the *ideal property*, also known as property (IP).

1 Introduction

The extend to which a C^*-algebra contains projections is decisive for its structure and properties. Abundance of projections can be expressed in many ways, several of which were proven to be equivalent by Brown and Pedersen in [6]. They refer to C^*-algebras satisfying these equivalent conditions as having *real rank zero*, written $\mathrm{RR}(-) = 0$, (where the real rank is a non-commutative notion of dimension). One of these equivalent conditions states that every non-zero hereditary sub-algebra has an approximate unit consisting of projections. Real rank zero is a non-commutative analog of being totally disconnected (because an abelian C^*-algebra $C_0(X)$, where X is a locally compact Hausdorff space, is of real rank zero if and only if X is totally disconnected). Another, weaker, condition, that we shall consider here is the ideal property (denoted (IP)) that projections in the C^*-algebra separate ideals.

The interest in C^*-algebras of real rank zero comes in parts from the fact that many C^*-algebras of interest happen—sometimes surprisingly—to be of real rank zero, and it comes in parts from Elliott's classification conjecture which predicts that separable nuclear C^*-algebras be classified by some invariant that includes K-theory (and in some special cases nothing more than K-theory!). The Elliott conjecture has a particularly nice formulation for C^*-algebras of real rank zero, it has been verified for a wide class of C^*-algebras of real rank zero, and the Elliott conjecture may still hold (in its original form) within this class of C^*-algebras (there are counterexamples to Elliott's conjecture in the non-real rank zero case).

If the Elliott conjecture holds for a certain class of C^*-algebras, then one can decide whether a specific C^*-algebra in this class is of real rank zero or not by looking at its

Elliott invariant. In the unital stably finite case, the Elliott conjecture predicts that a "nice" C^*-algebra A is of real rank zero if and only if the image of $K_0(A)$ in $\mathrm{Aff}(T(A))$ is dense, where $T(A)$ is the simplex of normalized traces on A. This has been verified in [20] in the case where A in addition is exact and tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$. In the presence of some weak divisibility properties on $K_0(A)$, the condition that $K_0(A)$ has dense image in $\mathrm{Aff}(T(A))$ can be replaced with the weaker condition that projections in A separate traces on A.

In the simple, purely infinite case, where there are no traces, real rank zero is automatic as shown by Zhang in [22]. This result is here generalized, assuming separability, to the non-simple case. We are forced to consider obstructions to real rank zero that do not materialize themselves in the simple case, including topological properties of the primitive ideal space and K-theoretical obstructions (as explained in the abstract).

The notion of being purely infinite was introduced by Cuntz, [8], in the simple case and extended to non-simple C^*-algebras by Kirchberg and the second named author in [12] (see Remark 2.6 for the definition). The study of purely infinite C^*-algebras was motivated by Kirchberg's classification of separable, nuclear, (strongly) purely infinite C^*-algebras up to stable isomorphism by an ideal related KK-theory. This classification result, although technically and theoretically powerful, is hard to apply in practice; however, it has the following beautiful corollary: Two separable nuclear C^*-algebras A and B are isomorphic after being tensored by $\mathcal{O}_2 \otimes \mathcal{K}$ if and only if their primitive ideal spaces are homeomorphic.

Suppose that A is a separable nuclear C^*-algebra whose primitive ideal space has a basis for its topology consisting of compact-open sets. Then, thanks to a result of Bratteli and Elliott, [4], there is an AF-algebra B whose primitive ideal space is homeomorphic to that of A. It follows that $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$; the latter C^*-algebra is of real rank zero, whence so is the former, whence so is $A \otimes \mathcal{O}_2$. In other words, if A is separable and nuclear, then $\mathrm{RR}(A \otimes \mathcal{O}_2) = 0$ if and only if the primitive ideal space of A has a basis of compact-open sets. Seeking to give a direct proof of this result and to drop the nuclearity hypothesis on A, we started the investigations leading to this article.

The paper is divided into three sections. In Section 2 we remind the reader of some of the relevant definitions and concepts, and it is shown that a purely infinite C^*-algebra has property (IP) if and only if its primitive ideal space has a basis of compact-open sets. Section 3 contains a discussion of the K-theoretical obstruction, that we call *K_0-liftable*, to having real rank zero and some technical ingredients that are needed for the proof of our main result, mostly related to lifting properties of projections. The final Section 4 contains our main result (formulated in the abstract) and some corollaries thereof.

Throughout this paper, the symbol $\otimes$ will mean the minimal tensor product of C^*-algebras; and by an ideal of an arbitrary C^*-algebra we will, unless otherwise specified, mean a closed and two-sided ideal.

2 Purely infinite C^*-algebras with property (IP)

In this section we show, among other things, that a purely infinite separable C^*-algebra has the ideal property if and only if its primitive ideal space has a basis consisting of compact-open sets. We begin by explaining the concepts that go into this statement.

Remark 2.1 (The ideal property (IP)) A C^*-algebra A has *the ideal property,* abbreviated (IP), if projections in A separate ideals in A, i.e., whenever I, J are ideals in A such that $I \nsubseteq J$, then there is a projection in $I \setminus J$.

The ideal property first appeared in Ken Stevens' Ph.D. thesis, where a certain class of (non-simple) C^*-algebras with the ideal property were classified by a K-theoretical invariant; later the first named author has studied this concept extensively, see e.g., [16] and [15].

Remark 2.2 (The primitive ideal space) The primitive ideal space, denoted $\mathrm{Prim}(A)$, of a C^*-algebra A is the set of all primitive ideals in A (e.g., kernels of irreducible representations) equipped with the Jacobsen topology. The Jacobsen topology is given as follows: if $\mathcal{M} \subseteq \mathrm{Prim}(A)$ and $J \in \mathrm{Prim}(A)$, then

$$J \in \overline{\mathcal{M}} \iff \bigcap_{I \in \mathcal{M}} I \subseteq J.$$

There is a natural lattice isomorphism between the ideal lattice, denoted $\mathrm{Ideal}(A)$, of A and the lattice, $\mathbb{O}(\mathrm{Prim}(A))$, of open subsets of $\mathrm{Prim}(A)$ given as

$$\begin{aligned} J \in \mathrm{Ideal}(A) \quad &\rightsquigarrow \quad \{I \in \mathrm{Prim}(A) \colon J \subseteq I\}^{\mathrm{c}} \in \mathbb{O}(\mathrm{Prim}(A)), \\ U \in \mathbb{O}(\mathrm{Prim}(A)) \quad &\rightsquigarrow \quad J = \bigcap_{I \in U^{\mathrm{c}}} I \in \mathrm{Ideal}(A), \end{aligned}$$

(where U^{c} denotes the complement of U). A subset of $\mathrm{Prim}(A)$ is said to be compact[1] if it has the Heine-Borel property. In the non-Hausdorff setting, compact sets need not be closed; compactness is preserved under forming finite unions, but not under (finite or infinite) intersections.

Subsets of $\mathrm{Prim}(A)$ which are both compact and open are, naturally, called compact-open. An ideal J in A corresponds to a compact-open subset in $\mathrm{Prim}(A)$ if and only if it has the following property (which is a direct translation of the Heine-Borel property): Whenever $\{J_\alpha\}$ is an increasing net of ideals in A such that $J = \overline{\bigcup_\alpha J_\alpha}$, then $J = J_\alpha$ for some α. We shall often—sloppily—refer to such ideals as *compact ideals.*

We are particularly interested in the case where $\mathrm{Prim}(A)$ has a basis (for its topology) consisting of compact-open sets. When $\mathrm{Prim}(A)$ is locally compact and Hausdorff this is the case precisely when $\mathrm{Prim}(A)$ is totally disconnected (all connected components are

[1]Some authors would rather call such a space *quasi-compact* and reserve the term "compact" for spaces that also are Hausdorff.

singletons). In general, $\mathrm{Prim}(A)$ has a basis of compact-open sets if and only if every (non-empty) open subset is the union of an increasing net of compact-open sets, or, equivalently, if and only if every ideal J in A is equal to $\overline{\bigcup_\alpha J_\alpha}$ for some increasing net $\{J_\alpha\}_\alpha$ of compact ideals.

If $\mathrm{Prim}(A)$ is finite, which happens precisely when $\mathrm{Ideal}(A)$ is finite, then all subsets are compact, whence $\mathrm{Prim}(A)$ has a basis of compact-open sets. The space $\mathrm{Prim}(A)$ is totally disconnected in this case if and only if it is Hausdorff; or, equivalently, if and only if A is the direct sum of finitely many simple C^*-algebras. See also Example 4.8.

If A is a separable C^*-algebra, then $\mathrm{Prim}(A)$ is a locally compact second countable T_0-space in which every (closed) prime[2] subset is the closure of a point. Conversely, if X is a space with these properties, and if X has a basis for its topology consisting of compact-open sets, then X is homeomorphic to $\mathrm{Prim}(A)$ for some separable AF-algebra A, as shown by Bratteli and Elliott in [4].

We shall need the following (probably well-known) easy lemma:

Lemma 2.3 *Let A be a C^*-algebra, let I, I_1, I_2 be ideals in A, and let $\pi\colon A \to A/I$ be the quotient mapping.*

(i) *If I_1 and I_2 are compact, then so is $I_1 + I_2$.*

(ii) *If I is compact and if J is a compact ideal in A/I, then $\pi^{-1}(J)$ is compact.*

Proof: (i). The union of two compact sets is again compact (also in a T_0-space).

(ii). Let $\{K_\alpha\}_\alpha$ be an arbitrary upwards directed family of ideals in A such that $\bigcup_\alpha K_\alpha$ is dense in $\pi^{-1}(J)$. Then $J = \overline{\bigcup_\alpha \pi(K_\alpha)}$, whence $J = \pi(K_{\alpha_1})$ for some α_1. As I is contained in $\pi^{-1}(J)$, it is equal to the closure of $\bigcup_\alpha (I \cap K_\alpha)$, whence $I = I \cap K_{\alpha_2}$ for some α_2. It now follows that $\pi^{-1}(J) = K_\alpha$ whenever α is greater than or equal to both α_1 and α_2. $\square$

We shall show later (in Corollary 4.4) that the class of C^*-algebras, for which the primitive ideal space has a basis of compact-open sets, is closed under extensions.

Remark 2.4 (Scaling elements) Scaling elements were introduced by Blackadar and Cuntz in [2] as a mean to show the existence of projections in simple C^*-algebras that admit no dimension function. An element x in a C^*-algebra A is called a scaling element if x is a contraction and x^*x is a unit for xx^*, ie., if $x^*xxx^* = xx^*$. Blackadar and Cuntz remark that if x is a scaling element, then $v = x + (1 - x^*x)^{1/2}$ is an isometry in the unitization of A, whence $p = 1 - vv^*$ is a projection in A. Moreover, if a is a positive element in A such that $x^*xa = a$ and $xx^*a = 0$, then $pa = a$. In this way we get a "lower bound" on the projection p.

Remark 2.5 (Cuntz' comparison theory) We recall briefly the notion of comparison of positive elements in a C^*-algebra A, due to Cuntz, [7]. Given $a, b \in A^+$, write $a \precsim b$ if

[2] A set F is called prime if whenever G and H are closed sets with $F = G \cup H$, then $F = G$ or $F = H$.

for all $\varepsilon > 0$ there is $x \in A$ such that $\|x^*bx - a\| < \varepsilon$. Let $(a-\varepsilon)_+$ denote the element obtained by applying the function $t \mapsto \max\{t-\varepsilon, 0\}$ to a. It is shown in [19] that if a, b are positive elements in A and if $\varepsilon > 0$, then $\|a - b\| < \varepsilon$ implies $(a-\varepsilon)_+ \precsim b$; and $a \precsim b$ and $a_0 \in \overline{(a-\varepsilon)_+ A (a-\varepsilon)_+}$ implies that $a_0 = x^*bx$ for some $x \in A$.

We shall also need the following fact: If $a \precsim b$ and $\varepsilon > 0$, then there exists a contraction $z \in A$ such that $z^*z(a-\varepsilon)_+ = (a-\varepsilon)_+$ and $zz^* \in \overline{bAb}$. Indeed, there is a positive contraction e in $\overline{(a-\varepsilon/2)_+ A (a-\varepsilon/2)_+}$ such that $e(a-\varepsilon)_+ = (a-\varepsilon)_+$, and by the result mentioned above there is $x \in A$ such that $e = x^*bx$. The element $z = b^{1/2}x$ is now as desired.

Remark 2.6 (Purely infinite C^*-algebras) A (possibly non-simple) C^*-algebra A is said to be *purely infinite* if A has no character (or, equivalently, no abelian quotients) and if

$$\forall a, b \in A^+ : a \in \overline{AbA} \iff a \precsim b,$$

where $\overline{AbA}$ denotes the ideal in A generated by the element b. Observe that the implication "$\Leftarrow$" above is trivial and holds for all C^*-algebras.

It is shown in [12] that any positive element a in a purely infinite C^*-algebra is properly infinite (meaning that $a \oplus a \precsim a \oplus 0$ in $M_2(A)$); and in particular, all (non-zero) projections in a purely infinite C^*-algebra are properly infinite (in the standard sense: $p \in A$ is properly infinite if there are projections $p_1, p_2 \in A$ such that $p_j \le p$, $p_1 \perp p_2$, and $p_1 \sim p_2 \sim p$).

It is also proved in [12] that $A \otimes \mathcal{O}_\infty$ and $A \otimes \mathcal{O}_2$ are purely infinite for all C^*-algebras A, and hence that $A \otimes B$ is purely infinite whenever B is a Kirchberg algebra[3] (because these satisfy $B \cong B \otimes \mathcal{O}_\infty$, see [11]).

Proposition 2.7 *Let I be an ideal in a separable purely infinite C^*-algebra A. Then the following conditions are equivalent:*

(i) *I corresponds to a compact-open subset of* $\mathrm{Prim}(A)$*, i.e., I is compact.*

(ii) *I is generated by a single projection in A.*

(iii) *I is generated by a finite family of projections in A.*

Proof: (iii) $\Rightarrow$ (i). Suppose that I is generated (as an ideal) by the projections $p_1, \dots, p_n$, and suppose that $\{I_\alpha\}_\alpha$ is an increasing net of ideals in A such that $\bigcup_\alpha I_\alpha$ is a dense (algebraic) ideal in I. Then $\bigcup_\alpha I_\alpha$ contains the projections $p_1, \dots, p_n$ (because it contains the Pedersen ideal[4] of I, and the Pedersen ideal of a C^*-algebra contains all projections of the C^*-algebra). It follows that $p_1, \dots, p_n$ belong to I_α for some α, whence $I = I_\alpha$.

(i) $\Rightarrow$ (ii). By separability of A (and hence of I), I contains a strictly positive element, and is hence generated (as an ideal) by a single positive element a. For each $\varepsilon \ge 0$ let I_ε be the ideal in A generated by $(a-\varepsilon)_+$. Then $I = \overline{\bigcup_{\varepsilon>0} I_\varepsilon}$, so by assumption (and Remark 2.2), $I = I_{\varepsilon_0}$ for some $\varepsilon_0 > 0$. It follows in particular that $a \precsim (a-\varepsilon_0)_+$, cf. Remark 2.6.

[3] A simple, separable, nuclear, purely infinite C^*-algebra.

[4] This is the smallest dense algebraic two-sided ideal in the C^*-algebra.

Choose ε_1 such that $0 < \varepsilon_1 < \varepsilon_0$. As A is purely infinite, all its positive elements, and in particular $(a-\varepsilon_1)_+$, are properly infinite (see [12, Definition 3.2]). Use [12, Proposition 3.3] (and [12, Lemma 2.5 (i)]) to find mutually orthogonal positive elements b_1, b_2 in $\overline{(a-\varepsilon_1)_+ I (a-\varepsilon_1)_+}$ such that $(a-\varepsilon_0)_+ \precsim b_1$ and $(a-\varepsilon_0)_+ \precsim b_2$. Then $a \precsim b_1$ (a relation that also holds relatively to I) and $a \precsim b_2$ (whence b_2 is full in I).

By Remark 2.5 there is $x \in I$ such that $x^*x(a-\varepsilon_1)_+ = (a-\varepsilon_1)_+$ and xx^* belongs to $\overline{b_1 I b_1} \subseteq \overline{(a-\varepsilon_1)_+ A (a-\varepsilon_1)_+}$. We conclude that x is a scaling element which satisfies $x^*xb_2 = b_2$ and $xx^*b_2 = 0$. By the result of Blackadar and Cuntz mentioned in Remark 2.4 above there is a projection $p \in I$ such that $pb_2 = b_2$. As b_2 is full in I, so is p.

(ii) $\Rightarrow$ (iii) is trivial. $\square$

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) do not require separability of A.

The corollary below follows immediately from Proposition 2.7 (and from Remark 2.2).

Corollary 2.8 *Let A be a separable purely infinite C^*-algebra where* $\mathrm{Prim}(A)$ *has a basis of compact-open sets. Then any ideal in A is either generated by a single projection or is the closure of the union of an increasing net of ideals each of which is generated by a single projection.*

Converserly, if A is any C^*-algebra (not necessarily separable or purely infinite), and if any ideal in A either is generated by a single projection or is the closure of the union of an increasing net of ideals with this property, then $\mathrm{Prim}(A)$ *has a basis for its topology consisting of compact-open sets.*

Lemma 2.9 *Let A be a purely infinite C^*-algebra and let B be a hereditary sub-C^*-algebra of A. Then each projection in $\overline{ABA}$, the ideal in A generated by B, is equivalent to a projection in B.*

Proof: Let p be a projection in $\overline{ABA}$. The family of ideals in A generated by a single positive element in B is upwards directed (if I_1 is generated by $b_1 \in B^+$ and I_2 is generated by $b_2 \in B^+$, then $I_1 + I_2$ is generated by $b_1 + b_2 \in B^+$). The union of these ideals is dense in I and therefore contains p. It follows that p belongs to $\overline{AbA}$ for some $b \in B^+$. As A is purely infinite, $p \precsim b$, whence $p = z^*bz$ for some $z \in A$ (because p is a projection, cf. [19, Proposition 2.7]). Put $v = b^{1/2}z$. Then $v^*v = p$, and $vv^* \in B$ is therefore a projection which is equivalent to p. $\square$

Proposition 2.10 *Any hereditary sub-C^*-algebra of a purely infinite C^*-algebra with property (IP) again has property (IP).*

Proof: Let A_0 be a hereditary sub-C^*-algebra of a purely infinite C^*-algebra A with property (IP), and let I_0 and J_0 be ideals in A_0 with $I_0 \not\subseteq J_0$. Let I and J be the ideals in A generated by I_0 and J_0, respectively. Then $I \not\subseteq J$, and so, by assumption, there is a projection $p \in I \setminus J$. By Lemma 2.9, p is equivalent to a projection $p' \in I_0$; and p' does not belong J, and hence not to J_0. $\square$

Condition (iii) below was considered by Brown and Pedersen in [5, Theorem 3.9 and Discussion 3.10] and was there given the name *purely properly infinite*. Brown and Pedersen noted that purely properly infinite C^*-algebras are purely infinite (in the sense discussed in Remark 2.6). Brown kindly informed us that this property is equivalent with properties (i) and (ii) below. We thank Larry Brown for allowing us to include this statement here.

Proposition 2.11 *The following four conditions are equivalent for any separable C^*-algebra A.*

(i) *A is purely infinite and* $\mathrm{Prim}(A)$ *has a basis for its topology consisting of compact-open sets.*

(ii) *A is purely infinite and has property (IP).*

(iii) *Any non-zero hereditary sub-C^*-algebras of A is generated as an ideal by its properly infinite projections.*

(iv) *Every non-zero hereditary sub-C^*-algebra in any quotient of A contains an infinite projection.*

The implications (i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) *hold also when A is non-separable.*

Proof: Separability is assumed only in the proof of "(i) $\Rightarrow$ (ii)".

(ii) $\Rightarrow$ (i). Let I be an ideal in A. Then I is generated by its projections (because A has property (IP)). Let Λ be the net of finite subsets of the set of projections in I, and, for each $\alpha \in \Lambda$, let I_α be the ideal in A generated by the projections in the finite set α. Then I_α is compact (by Proposition 2.7), and $\bigcup_{\alpha\in\Lambda} I_\alpha$ is dense in I. This shows that $\mathrm{Prim}(A)$ has a basis of compact-open sets, cf. Remark 2.2.

(i) $\Rightarrow$ (ii). Suppose that (i) holds, and let I, J be ideals in A such that $I \nsubseteq J$. From Corollary 2.8 there is an increasing net of ideals I_α in A each generated by a single projection, say p_α, such that $\bigcup_\alpha I_\alpha$ is dense in I. Now, $I_\alpha \nsubseteq J$ for some α, and so the projection p_α belongs to $I \setminus J$.

(ii) $\Rightarrow$ (iii). Every non-zero projection in a purely infinite C^*-algebra is properly infinite (see Remark 2.6 or [12, Theorem 4.16]) and so it suffices to show that any hereditary sub-C^*-algebra of A has property (IP); but this follows from Proposition 2.10 and the assumption that A is purely infinite and has property (IP).

(iii) $\Rightarrow$ (iv). Let I be an ideal in A, and let B be a non-zero hereditary sub-C^*-algebra of A/I. Let $\pi\colon A \to A/I$ denote the quotient mapping. By (iii) and Lemma 2.9 there is a properly infinite projection p in $\pi^{-1}(B) \setminus I$; and so $\pi(p)$ is a non-zero properly infinite (and hence infinite) projection in B.

(iv) $\Rightarrow$ (i). It follows from [12, Proposition 4.7] that A is purely infinite. We must show that $\mathrm{Prim}(A)$ has a basis of compact-open sets. We use the equivalent formulation given in Remark 2.2, see also Corollary 2.8.

Let I be an ideal in A, and let $\{I_\alpha\}$ be the family of all compact ideals contained in I. Then $\{I_\alpha\}_\alpha$ is upwards directed (by Lemma 2.3 (i)). Let I_0 be the closure of the union

of the ideals I_α. We must show that $I_0 = I$. Suppose, to reach a contradiction, that $I_0 \subset I$. Then, by (iv), I/I_0 contains a non-zero projection p. The projection p lifts to a projection q in I/I_α for some α (by semiprojectivity of the C^*-algebra $\mathbb{C}$, see also the proof of Lemma 4.1 below). Let J be the ideal in I/I_α generated by the projection q. Then J is compact, whence so is its pre-image $I' \subseteq I$ under the quotient mapping $I \to I/I_\alpha$, cf. Lemma 2.3 (ii). As the image of I' under the quotient mapping $I \to I/I_0$ contains the projection p we conclude that I' is not contained in I_0, which is in contradiction with the construction of I_0. $\square$

Property (i) in the lemma below is pretty close to saying that the hereditary sub-C^*-algebra $\overline{aAa}$ has an approximate unit consisting of projections, and hence that A is of real rank zero. In fact, if A has stable rank one (which by the way never can happen when A is purely infinite and not stably projectionless!), then property (i) below would have implied that A has real rank zero. In the absence of stable rank one we get real rank zero from condition (i) below if a K-theoretical condition, discussed in the next section, is satisfied.

Lemma 2.12 *Let A be a purely infinite C^*-algebra with property (IP).*

(i) *For each positive element $a \in A$ and for each $\varepsilon > 0$, there is a projection $p \in \overline{aAa}$ such that $(a-\varepsilon)_+ \precsim p$.*

(ii) *For each element $x \in A$ and for each $\varepsilon > 0$, there is a projection $p \in A$ and an element $y \in A$ such that $\|x - y\| \le \varepsilon$ and $y \in \overline{ApA}$.*

Proof: (i). The hereditary C^*-algebra $\overline{aAa}$ is purely infinite and has property (IP) (by Lemma 2.9). We can therefore apply Corollary 2.8 to $\overline{aAa}$ to obtain an increasing net $\{I_\alpha\}_\alpha$ of ideals in $\overline{aAa}$ each generated by a single projection such that $\bigcup_\alpha I_\alpha$ is a dense algebraic ideal in $\overline{aAa}$. It follows that $(a-\varepsilon)_+$ belongs to $\bigcup_\alpha I_\alpha$, and hence to I_α for some α. Let p be a projection that generates the ideal I_α. Then $(a-\varepsilon)_+ \precsim p$, because $(a-\varepsilon)_+$ belongs to the ideal generated by p.

(ii). Write $x = v|x|$ with v a partial isometry in A^{**}, and put $y = v(|x|-\varepsilon)_+ \in A$. Then $\|x-y\| \le \varepsilon$ and $|y| = (|x|-\varepsilon)_+$. Use (i) to find a projection p in A such that $|y| \precsim p$. Then $|y|$, and hence also y, belong to $\overline{ApA}$. $\square$

We continue this section with a general result on C^*-algebras (not necessarily purely infinite) with property (IP) that is relevant for the discussion in Section 3.

Proposition 2.13 *Any separable stable C^*-algebra with property (IP) has an approximate unit consisting of projections.*

Proof: If A is a separable stable C^*-algebra containing a full projection p, then A is isomorphic to $pAp \otimes \mathcal{K}$ by Brown's theorem; and so in particular A has an approximate unit consisting of projections.

Suppose that A is separable, stable and with property (IP). Then $A = \overline{\bigcup_\alpha A_\alpha}$ for some increasing net $\{A_\alpha\}_\alpha$ of ideals in A each of which is generated by a finite set of projections,

cf. the proof of "(i) $\Rightarrow$ (ii)" in Proposition 2.11. We claim that each A_α is in fact generated by a single projection. Indeed, suppose that A_α is generated as an ideal by the projections $p_1, p_2, \dots, p_n$; then $p_1 \oplus p_2 \oplus \cdots \oplus p_n$ is equivalent to (or equal to) a projection $p \in A_\alpha$, because A_α is stable (being an ideal in a stable C^*-algebra). It follows that A_α is generated by the projection p. By the first part of the proof, A_α has an approximate unit consisting of projections. As this holds for all α we conclude that also A has an approximate unit consisting of projections. $\square$

Proposition 2.14 below was shown in [13] by Kirchberg and the second named author for C^*-algebras of the real rank zero. We extend here this result to the broader class of C^*-algebras with property (IP). We refer to [13] for the definitions of being strongly, respectively, weakly purely infinite.

Proposition 2.14 *Let A be a C^*-algebra with property (IP). The following are equivalent:*

(i) *A is purely infinite.*

(ii) *A is strongly purely infinite.*

(iii) *A is weakly purely infinite.*

Proof: (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) are (trivially) true for all C^*-algebras A (see [13, Theorem 9.1]).

(i) $\Rightarrow$ (ii). It follows from Lemma 2.12 and from [13, Remark 6.2] (see also the proof of [13, Proposition 6.3]) that any C^*-algebra with property (IP) has the *locally central decomposition property*; and [13, Theorem 6.8] says that any purely infinite C^*-algebra with the locally central decomposition property is strongly purely infinite.

(iii) $\Rightarrow$ (i). Assume that A is weakly purely infinite. By the comment following [13, Proposition 4.18], A is purely infinite if every quotient of A has the property (SP) (i.e., each non-zero hereditary sub-C^*-algebra contains a non-zero projection). Both property (IP) and weak pure infiniteness pass to quotients, cf. [13, Proposition 4.5], so it will be enough to prove that any non-zero hereditary sub-C^*-algebra B of A contains a non-zero projection.

As A is weakly purely infinite, it is pi-n for some natural number n (see [13, Definition 4.3]). By the Glimm lemma (see [12, Proposition 4.10]) there is a non-zero *-homomorphism from $M_n(C_0((0,1]))$ into B. So we get non-zero pairwise equivalent and orthogonal positive elements $e_1, \dots, e_n$ in B. The ideal in A generated by e_1 contains a non-zero projection p. As A is assumed to be pi-n we can use [13, Lemma 4.7] to conclude that $p \precsim e_1 \otimes 1_n$; and as $e_1 \otimes 1_n \precsim e_1 + e_2 + \cdots + e_n =: b \in B$ (see [12, Lemma 2.8]) it follows from [12, Proposition 2.7 (iii)] that p is equivalent to a (necessarily non-zero) projection q in $\overline{bAb} \subseteq B$. (It has been used twice above that $p \precsim (1-\varepsilon)p = (p-\varepsilon)_+$ when p is a projection and $0 \leq \varepsilon < 1$.) $\square$

3 Lifting projections

We consider here when projections in a quotient of a purely infinite C^*-algebra lift to the C^*-algebra itself. We begin with a discussion of a K-theoretical obstruction to lifting projections:

Definition 3.1 A C^*-algebra A is said to be K_0-*liftable* if for every pair of ideals $I \subset J$ in A, the extension

$$0 \longrightarrow I \xrightarrow{\iota} J \xrightarrow{\pi} J/I \longrightarrow 0$$

has the property that $K_0(\pi)\colon K_0(J) \to K_0(J/I)$ is surjective (or, equivalently, that the index map $\delta\colon K_0(J/I) \to K_1(I)$ is zero, or, equivalently, if the induced map $K_1(\iota)\colon K_1(I) \to K_1(J)$ is injective).

As pointed out to us by Larry Brown, it suffices to check K_0-liftability for $J = A$ (i.e., A is K_0-liftable if and only if the induced map $K_0(A) \to K_0(A/I)$ is onto for every ideal I in A), because if $K_1(I) \to K_1(A)$ is injective, then so is $K_1(I) \to K_1(J)$ whenever $I \subseteq J \subseteq A$.

Every simple C^*-algebra is automatically K_0-liftable (there are no non-trivial sequences $0 \to I \to J \to J/I \to 0$ for ideals $I \subset J$ in a simple C^*-algebra).

The property real rank zero passes from a C^*-algebra to its ideals (cf. Brown and Pedersen, [6]), and in the same paper it is shown that the map $K_0(A) \to K_0(A/I)$ is onto whenever A is a C^*-algebra of real rank zero and I is an ideal in A. Hence all C^*-algebras of real rank zero are K_0-liftable.

Being K_0-liftable passes to hereditary sub-C^*-algebras:

Lemma 3.2 *Any hereditary sub-C^*-algebra of a separable K_0-liftable C^*-algebra is again K_0-liftable.*

Proof: Let A be a separable K_0-liftable C^*-algebra, and let A_0 be a hereditary sub-C^*-algebra of A. Let $I_0 \subset J_0$ be ideals in A_0, and let $I \subset J$ be the ideals in A generated by I_0 and J_0, respectively.

Then J_0 is a full hereditary sub-C^*-algebra of J, and (the image in J/I of) J_0/I_0 is a full hereditary sub-C^*-algebra in J/I. The commutative diagram

$$\begin{array}{ccc} J_0 & \longrightarrow & J_0/I_0 \\ \downarrow & & \downarrow \\ J & \longrightarrow & J/I \end{array}$$

induces a commutative diagram of K_0-groups

$$\begin{array}{ccc} K_0(J_0) & \longrightarrow & K_0(J_0/I_0) \\ \downarrow & & \downarrow \\ K_0(J) & \longrightarrow & K_0(J/I), \end{array}$$

where the vertical maps are isomorphisms (by stability of K_0 and by Brown's theorem) and the lower horizontal map is surjective by assumption. Hence the upper horizontal map $K_0(J_0) \to K_0(J_0/I_0)$ is surjective. □

The next lemma expresses when an extension of two K_0-liftable C^*-algebras is K_0-liftable:

Lemma 3.3 *Let*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

be a short-exact sequence of C^*-algebras. Then A is K_0-liftable if and only if I and B are K_0-liftable and the induced map $K_0(A) \to K_0(B)$ is onto.

Proof: "If". We use the remark below Definition 3.1 whereby it suffices to show that the map $K_0(A) \to K_0(A/J)$ is onto whenever J is an ideal in A. To this end, consider the diagram of C^*-algebras with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \cap J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & I/I \cap J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A/J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi(J) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/\pi(J) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

that induces the following diagram at the level of K_0:

$$\begin{tikzcd}
K_0(I) \arrow[r, "*"] \arrow[d] & K_0(I/I \cap J) \arrow[d] \\
K_0(A) \arrow[r] \arrow[d, "*"'] & K_0(A/J) \arrow[d] \\
K_0(B) \arrow[r, "*"] & K_0(B/\pi(J))
\end{tikzcd}$$

where the vertical sequences are exact and the maps marked with an asterisk are surjective (by our assumptions). A standard diagram chase shows that the map $K_0(A) \to K_0(A/J)$ is surjective.

"Only if". If A is K_0-liftable, then clearly so is I, and $K_0(A) \to K_0(B)$ is onto. We proceed to prove that B is K_0-liftable. Let $J \subset L$ be ideals in B, and consider the commuting diagram

$$\begin{tikzcd}
\pi^{-1}(L) \arrow[r] \arrow[d] & L \arrow[d] \\
\pi^{-1}(L)/\pi^{-1}(J) \arrow[r, "\cong"] & L/J,
\end{tikzcd}$$

that induces the commuting diagram

$$\begin{tikzcd}
K_0(\pi^{-1}(L)) \arrow[r] \arrow[d] & K_0(L) \arrow[d] \\
K_0\big(\pi^{-1}(L)/\pi^{-1}(J)\big) \arrow[r, "\cong"] & K_0(L/J).
\end{tikzcd}$$

The left-most vertical map is onto by K_0-liftability of A, which implies surjectivity of the right-most vertical map. $\square$

We proceed to describe when certain tensor products are K_0-liftable

Lemma 3.4 *The tensor product $A \otimes \mathcal{O}_2$ is K_0-liftable for every C^*-algebra A; and the tensor product $A \otimes \mathcal{O}_\infty$ is K_0-liftable if and only if A itself is K_0-liftable.*

Proof: If D is a simple nuclear C^*-algebra, then the mapping $I \mapsto I \otimes D$ defines a lattice isomorphism from $\mathrm{Ideal}(A)$ onto $\mathrm{Ideal}(A \otimes D)$ (surjectivity follows from a theorem of Blackadar, [1], see also [3, Proposition 2.16]). Moreover, by Blackadar's theorem or by exactness of D, if $I \subset J$ are ideals in A, then $(J \otimes D)/(I \otimes D)$ is isomorphic to $(J/I) \otimes D$. Hence, to prove K_0-liftability of $A \otimes D$ it suffices to show that the induced map $K_0(J \otimes D) \to K_0((J/I) \otimes D)$ is surjective, or, equivalently, that the index map $K_0((J/I) \otimes D) \to K_1(I \otimes D)$ is zero. The latter holds for all C^*-algebras A if $D = \mathcal{O}_2$ because $K_1(I \otimes \mathcal{O}_2) = 0$.

To prove the last statement, consider the commutative diagram

$$\begin{tikzcd}
J \arrow[r] \arrow[d] & J/I \arrow[d] \\
J \otimes \mathcal{O}_\infty \arrow[r] & (J/I) \otimes \mathcal{O}_\infty,
\end{tikzcd}$$

where the vertical maps are defined by $x \mapsto x \otimes 1$. It follows from the Künneth theorem that the vertical maps above induce isomorphisms at the level of K_0. It is now clear that $A \otimes \mathcal{O}_\infty$ is K_0-liftable if and only if A is K_0-liftable. $\square$

We now proceed with the projection lifting results. We need a sequence of lemmas.

Lemma 3.5 *Let A be a C^*-algebra, let x be an element in A, and suppose that there is a positive element e in A such that $x^*x \precsim e$, and x^*x and xx^* are orthogonal to e. Then x belongs to the closure of the invertible elements, $\mathrm{GL}(\widetilde{A})$, in the unitization $\widetilde{A}$ of A.*

Proof: Let $\varepsilon > 0$ be given. By the assumption that $|x|^2 = x^*x \precsim e$ and by Remark 2.6 we obtain a contraction $z \in A$ such that

$$(|x| - \varepsilon)_+ z^*z = (|x| - \varepsilon)_+, \qquad zz^* \in \overline{eAe}, \qquad zz^* \perp z^*z.$$

Now, $a = z + z^*$ is a self-adjoint contraction in A and $u = a + i\sqrt{1-a^2}$ is a unitary element in $\widetilde{A}$.

Write $x = v|x|$ with v a partial isometry in A^{**}, and put $x_\varepsilon = v(|x| - \varepsilon)_+ \in A$. Then $\|x - x_\varepsilon\| \le \varepsilon$,

$$x_\varepsilon u = v(|x|-\varepsilon)_+ u = v(|x|-\varepsilon)_+ a = v(|x|-\varepsilon)_+ z^*,$$

$z^* x_\varepsilon = 0$, and so

$$(x_\varepsilon u)^2 = v(|x|-\varepsilon)_+ z^* x_\varepsilon u = 0.$$

It follows that $x_\varepsilon + \lambda u^* = (x_\varepsilon u + \lambda 1)u^*$ is invertible in $\widetilde{A}$ for all $\lambda \neq 0$, whence x_ε belongs to the closure of $\mathrm{GL}(\widetilde{A})$. As $\varepsilon > 0$ was arbitrary, the lemma is proved. □

Lemma 3.6 *Let A be a C^*-algebra. Let x be an element in A and let e be a properly infinite projection in A such that x^*x is orthogonal to e and $x^*x \precsim e$. Then, for each $\varepsilon > 0$, there is a projection $p \in A$ such that $\|x - xp\| \le \varepsilon$.*

Proof: Because e is properly infinite (cf. Remark 2.6) there is a subprojection e_0 of e such that $e \precsim e_0$ and $e \precsim e - e_0$. As $|x|^2 = x^*x \precsim e \precsim e_0$ there is $z \in A$ with $z = e_0 z$ and

$$(|x| - \varepsilon/2)_+ = z^* e_0 z = z^* z$$

(see Remark 2.5). As zz^* and z^*z both are orthogonal to the projection $e - e_0$, and $z^*z \precsim e_0 \precsim e - e_0$, we conclude from Lemma 3.5 that z belongs to the closure of $\mathrm{GL}(\widetilde{A})$. By [18] there is a unitary u in $\widetilde{A}$ such that

$$u(|x|-\varepsilon)_+ u^* = u(z^*z - \varepsilon/2)_+ u^* = (zz^* - \varepsilon/2)_+ \in e_0 A e_0.$$

The projection $p = u^* e_0 u \in A$ thus satisfies $(|x|-\varepsilon)_+ p = (|x|-\varepsilon)_+$, which entails that

$$\|x(1-p)\| = \||x|(1-p)\| \le \varepsilon.$$

□

The lemma below and its proof are similar to [6, Lemma 3.13] and its proof.

Lemma 3.7 *Let A be a purely infinite C^*-algebra, let I be an ideal in A, and let B be a hereditary sub-C^*-algebra of A. Assume that I has property (IP). Let p be a projection in $B + I$ and assume that $B \cap pAp$ is full in A. Then there is a projection $q \in B$ such that $p - q$ belongs to I.*

Proof: Write $p = b + x$, with b a self-adjoint element in B and x a self-adjoint element in I. Take $\varepsilon > 0$ such that $2\|b\|\varepsilon + \varepsilon^2 < 1/2$. By Lemma 2.12 we can find an element $y \in I$ and a projection $f \in I$ such that $\|x - y\| < \varepsilon/2$ and such that y belongs to the ideal I_0 in I generated by f. By assumption, $B \cap pAp$ is full in A, so f is equivalent to a projection $g \in B \cap pAp$ (by Lemma 2.9). Put

$$b_1 = (1-g)b(1-g) + g \in B, \quad x_1 = (1-g)x(1-g) \in I, \quad y_1 = (1-g)y(1-g) \in I_0.$$

Then $p = b_1 + x_1$ and $\|x_1 - y_1\| \leq \|x - y\| < \varepsilon/2$. Now, p and g commute, g and py_1^2p belong to pI_0p, g is full in I_0, and $py_1^2p \perp g$. By pure infiniteness of A we deduce that $py_1^2p \precsim g$. We can now use Lemma 3.6 to conclude that there is a projection $r \in pI_0p \subseteq pIp$ such that $\|y_1(p-r)\| < \varepsilon/2$. Hence $\|x_1(p-r)\| < \varepsilon$.

Now,

$$\begin{aligned} p - r &= p^*(p-r)p \\ &= b_1(p-r)b_1 + b_1(p-r)x_1 + x_1(p-r)b_1 + x_1(p-r)x_1 \\ &= b_2 + x_2, \end{aligned}$$

where

$$b_2 = b_1(p-r)b_1 \in B, \qquad x_2 = b_1(p-r)x_1 + x_1(p-r)b_1 + x_1(p-r)x_1 \in I.$$

Note that

$$\begin{aligned} \|x_2\| &\leq \|b_1\|\|(p-r)x_1\| + \|x_1(p-r)\|\|b_1\| + \|x_1(p-r)\|^2 \\ &\leq 2\|b\|\|x_1(p-r)\| + \|x_1(p-r)\|^2 \\ &\leq 2\|b\|\varepsilon + \varepsilon^2 \ < \ 1/2, \end{aligned}$$

where it has been used that x_1 is self-adjoint. This shows that the distance from b_2 to the projection $p - r$ is less than $1/2$, whence $1/2$ is not in the spectrum of b_2. The function $f = 1_{[1/2,\infty)}$ restricts to a continuous function on $\mathrm{sp}(p-r)$ and on $\mathrm{sp}(b_2)$, whence

$$p - r = f(p-r) = f(b_2) + x_3$$

for some $x_3 \in I$. We can take q to be $f(b_2)$. $\square$

Lemma 3.8 *Let A be a separable purely infinite C^*-algebra with property (IP). Then*

$$K_0(A) = \{[p] : p \text{ is a projection in } A\}.$$

Proof: By Proposition 2.13 every element in $K_0(A)$ is represented by a difference $[p_0] - [q_0]$, where p_0, q_0 are projections in $A \otimes \mathcal{K}$. Upon replacing p_0 and q_0 with $p_0 \oplus q_0$ and $q_0 \oplus q_0$, respectively, we can assume that q_0 belongs to the ideal generated by p_0, whence $q_0 \sim q_1 \leq p_0$ for some projection q_1 by pure infiniteness of A. The projection $p_0 - q_1 \in A \otimes \mathcal{K}$ is equivalent to a projection $p \in A$ by Lemma 2.9; and $[p_0] - [q_0] = [p_0] - [q_1] = [p_0 - q_1] = [p]$. $\square$

Lemma 3.9 *Let*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

be an extension where A is a separable purely infinite C^*-algebra with property (IP). Let q be a projection in B such that $[q]$ belongs to $K_0(\pi)(K_0(A))$. Then A contains an ideal A_0, which is generated by a single projection, such that $q \in \pi(A_0)$ and $[q] \in K_0(\pi|_{A_0})(K_0(A_0))$.

Proof: By Corollary 2.8 there is an increasing net $\{A_\alpha\}_\alpha$ of ideals in A, each generated by a single projection, such that $\bigcup_\alpha A_\alpha$ is dense in A. By the assumption that $[q] \in K_0(\pi)(K_0(A))$, and by Lemma 3.8, there is a projection $r \in A$ such that $[\pi(r)] = [q]$. Now, $r \in A_{\alpha_1}$ and $q \in \pi(A_{\alpha_2})$ for suitable α_1 and α_2. We can therefore take A_0 to be A_α, when α is chosen greater than or equal to both α_1 and α_2. $\square$

In the lemma below we identify A with the upper left corner of $M_n(A)$, and thus view A as a hereditary sub-C^*-algebra of $M_n(A)$ for any $n \in \mathbb{N}$.

Lemma 3.10 *Let*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

be an extension where A is a separable purely infinite C^*-algebra with property (IP). Then a full projection q in B lifts to a projection in $A + M_4(I)$ if and only if $[q] \in K_0(\pi)(K_0(A))$.

Proof: The pre-image of $B \subseteq M_4(B)$ under the quotient mapping $\pi \otimes \mathrm{id}_{M_4} \colon M_4(A) \to M_4(B)$ is $A + M_4(I)$. Hence it suffices to show that q lifts to a projection $p \in M_4(A)$.

By Lemmas 3.8 and 3.9, possibly upon replacing A by an ideal in A, we can assume that A contains a full projection e and a (not necessarily full) projection p_1 such that $[\pi(p_1)] = [q]$ in $K_0(B)$. Since e is full and properly infinite there are mutually orthogonal subprojections e_0 and e_1 of e such that e_0 is full in A, $[e_0] = 0$ in $K_0(A)$, and $e_1 \sim p_1$. Set $p' = e_0 + e_1$. Then $[\pi(p')] = [q]$ in $K_0(B)$, and $\pi(p')$ and q are both full and properly infinite in B, so they are equivalent (by [8, Theorem 1.4]). It follows that $\pi(p')$ is homotopic to q inside $M_4(B)$; and by standard non-stable K-theory, see e.g. [21, Lemma 2.1.7, Proposition 2.2.6, and 1.1.6], we conclude that q lifts to a projection p in $M_4(A)$. $\square$

Using pure infiniteness of A one can improve the lemma above to get the lifted projection inside $A + M_2(I)$ (instead of in $A + M_4(I)$). However, one cannot always get the lift in $A + I$ as Example 3.12 below shows. First we state and prove our main lifting result for projections in purely infinite C^*-algebras with the ideal property:

Proposition 3.11 *Every separable, purely infinite, K_0-liftable C^*-algebra A with property (IP) has the following projection lifting property: For any hereditary sub-C^*-algebra A_0 of A and for any ideal I_0 in A_0, every projection in the quotient A_0/I_0 lifts to a projection in A_0.*

Proof: Let A_0 and I_0 be as above, and let q be a projection in A_0/I_0. We must show that q lifts to a projection in A_0. Let $\pi \colon A_0 \to A_0/I_0$ denote the quotient mapping. Upon passing to a hereditary sub-C^*-algebra of A_0 (the pre-image $\pi^{-1}(q(A_0/I_0)q)$) we can assume that q is full in A_0/I_0 (and even that q is the unit for A_0/I_0). By Lemma 3.9 (and Proposition 2.10), possibly upon replacing A_0 with an ideal of A_0, we can further assume that A_0 contains a full projection, say g (and that $q \in \pi(A_0)$).

Put $A_{00} = (1-g)A_0(1-g)$ and $I_{00} = A_{00} \cap I_0$. It follows from Lemma 3.2 and our assumption that the map $K_0(A_{00}) \to K_0(A_{00}/I_{00})$ is onto. We can now use Lemma 3.10

to lift $q - \pi(g)$ to a projection p' in $A_{00} + M_4(I_{00})$. Thus $p'' = p' + g \in A_0 + M_4(I_0)$ is a lift of q, and $gA_0g \subseteq p''M_4(A_0)p'' \cap A_0$ is full in A_0. As A is assumed to have property (IP) we obtain from Proposition 2.10 that I_0 has property (IP), and so we can use Lemma 3.7 to get a projection $p \in A_0$ such that $p - p'' \in M_4(I_0)$; and p is a lift of q. $\square$

Example 3.12 Consider the C^*-algebra

$$A = \{f \in C([0,1], \mathcal{O}_2) : f(1) = sf(0)s^*\},$$

where $s \in \mathcal{O}_2$ is any non-unitary isometry. Let $\pi\colon A \to \mathcal{O}_2$ be given by $\pi(f) = f(0)$. Then we have a short exact sequence

$$0 \longrightarrow C_0((0,1), \mathcal{O}_2) \longrightarrow A \xrightarrow{\pi} \mathcal{O}_2 \longrightarrow 0.$$

The map $K_0(A) \to K_0(\mathcal{O}_2)$ is surjective, because $K_0(\mathcal{O}_2) = 0$. (One can show that $A \cong A \otimes \mathcal{O}_2$, and hence that A is K_0-liftable, cf. Lemma 3.4.) However, the unit $1 \in \mathcal{O}_2$ does not lift to a projection in A, because 1 is not homotopic to $ss^* \neq 1$ inside $\mathcal{O}_2$.

Of course, the ideal $C_0((0,1), \mathcal{O}_2)$ does not have property (IP), so this example does not contradict Proposition 3.11. But the example does show that Proposition 3.11 is false without the assumption that A (and hence the ideal I_{00}) has property (IP), and it shows that Lemma 3.10 does not hold with $A + M_4(I)$ replaced with $A + I$.

4 The main result

Here we state and prove our main result described in the abstract. Let us set up some notation.

Let $g_\varepsilon\colon \mathbb{R}^+ \to \mathbb{R}^+$ be the continuous function

$$g_\varepsilon(t) = \begin{cases} (\varepsilon - t)/\varepsilon, & t \leq \varepsilon \\ 0, & t \geq \varepsilon \end{cases}.$$

If A is a non-unital C^*-algebra and a is a positive element in A, then $g_\varepsilon(a)$ belongs to the unitization of A, but not to A. However,

$$I_\varepsilon(a) := \overline{Ag_\varepsilon(a)A}$$

is an ideal in A, and

$$H_\varepsilon(a) := \overline{g_\varepsilon(a)Ag_\varepsilon(a)}$$

is a hereditary subalgebra of A. The hereditary sub-C^*-algebra $H_\varepsilon(a)$ is full in $I_\varepsilon(a)$, i.e., $I_\varepsilon(a) = \overline{AH_\varepsilon(a)A}$.

The quotient C^*-algebra $A/I_\varepsilon(A)$ is unital and $a + I_\varepsilon(a)$ is invertible in $A/I_\varepsilon(a)$ (provided that $I_\varepsilon(a)$ is different from A). Indeed, $h(a) + I_\varepsilon(a)$ is a unit for $A/I_\varepsilon(a)$ and $f(a) + I_\varepsilon(a)$ is the inverse to $a + I_\varepsilon(a)$, when

$$h(t) = \begin{cases} \varepsilon^{-1}t, & t \leq \varepsilon \\ 1, & t \geq \varepsilon \end{cases}, \qquad f(t) = \begin{cases} \varepsilon^{-2}t, & t \leq \varepsilon \\ 1/t, & t \geq \varepsilon \end{cases}.$$

Lemma 4.1 *Assume that A is a separable purely infinite C^*-algebra whose primitive ideal space has a basis of compact-open sets. Let a be a positive element in A and let $\varepsilon > 0$. Assume that $I_\varepsilon(a) \neq A$. Then there is a projection e in $H_\varepsilon(a)$ and an ideal I in A such that $I = \overline{AeA} \subseteq I_\varepsilon(a)$, and A/I contains a projection f which is a unit for the element $(a-\varepsilon)_+ + I$ in A/I.*

Proof: If $I_\varepsilon(a)$ itself were compact, i.e., generated by a single projection, then, by Lemma 2.9, it would be generated by a projection $e \in H_\varepsilon(a)$. We could then take I to be $I_\varepsilon(a)$ and the projection f to be the unit of $A/I_\varepsilon(a)$.

Let us now consider the general case, where $I_\varepsilon(a)$ need not be compact. Find an increasing net of ideals I_α in A, each of which is generated by a single projection, such that $\bigcup_\alpha I_\alpha$ is dense in $I_\varepsilon(a)$, cf. Corollary 2.8. Then, for each α, we have a commutative diagram:

$$\begin{tikzcd} & A/I_\alpha \arrow[d, "\nu_\alpha"] \\ A \arrow[ur, "\pi_\alpha"] \arrow[r, "\pi"'] & A/I_\varepsilon(a) \end{tikzcd}$$

and $\|\pi_\alpha(x)\| \to \|\pi(x)\|$ for all $x \in A$. We saw above that $\pi(h(a))$ is a unit for $A/I_\varepsilon(a)$; so

$$\lim_\alpha \|\pi_\alpha\big(h(a) - h(a)^2\big)\| = \|\pi\big(h(a) - h(a)^2\big)\| = 0.$$

We can therefore take α such that $\|\pi_\alpha\big(h(a) - h(a)^2\big)\| < 1/4$, in which case $1/2$ does not belong to the spectrum of $\pi_\alpha(h(a))$.

The ideal I_α is by assumption generated by a projection, say g; and as g belongs to $I_\varepsilon(a)$ it is equivalent to a projection $e \in H_\varepsilon(a)$ by Lemma 2.9; whence $I := I_\alpha$ is generated by e.

The characteristic function $1_{[1/2,\infty)}$ is continuous on the spectrum of $\pi_\alpha(h(a))$; and it extends to a continuous function $\varphi\colon \mathbb{R}^+ \to [0,1]$ which satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$. Put

$$f = 1_{[1/2,\infty)}\big(\pi_\alpha(h(a))\big) = \pi_\alpha\big((\varphi \circ h)(a)\big) \in A/I.$$

Then f is a projection, and as $(\varphi \circ h)(a)\cdot(a-\varepsilon)_+ = (a-\varepsilon)_+$, we have

$$f\cdot\pi_\alpha\big((a-\varepsilon)_+\big) = \pi_\alpha\big((\varphi\circ h)(a)\cdot(a-\varepsilon)_+\big) = \pi_\alpha\big((a-\varepsilon)_+\big),$$

as desired. □

Theorem 4.2 *Let A be a separable purely infinite C^*-algebra. Then the real rank of A is zero if and only if A is K_0-liftable (cf. Definition 3.1) and the primitive ideal space of A has a basis for its topology consisting of compact-open sets.*

Proof: If $\mathrm{RR}(A) = 0$, then A has property (IP), whence $\mathrm{Prim}(A)$ has a basis consisting of compact-open sets, cf. Proposition 2.11. As remarked below Definition 3.1, it follows from [6] that every C^*-algebra of real rank zero is K_0-liftable. This proves the "only if" part.

We proceed to prove the "if" part, and so we assume that A is K_0-liftable and that $\mathrm{Prim}(A)$ has a basis of compact-open sets. Then, by Proposition 2.11, A has property (IP).

To show that $\mathrm{RR}(A) = 0$ we show that each hereditary sub-C^*-algebra of A has an approximate unit consisting of projections. Hereditary sub-C^*-algebras of purely infinite C^*-algebras are again purely infinite (see [12]), and it follows from Lemma 3.2 and Proposition 2.10 that any hereditary sub-C^*-algebra of A is K_0-liftable and has property (IP). Upon replacing a hereditary sub-C^*-algebra of A by A itself, it suffices to show that A has an approximate unit consisting of projections. To this end it suffices to show that, given a positive element a in A and $\varepsilon > 0$, there is a projection p in A such that $\|a - ap\| \le 3\varepsilon$.

Let $I_\varepsilon(a)$ and $H_\varepsilon(a)$ be as defined above Lemma 4.1. Then, as already observed, $H_\varepsilon(a)$ is a full hereditary sub-C^*-algebra of $I_\varepsilon(a)$; and $\|ae\| \le \varepsilon\|e\|$ for all $e \in H_\varepsilon(a)$ by construction of $H_\varepsilon(a)$.

Suppose that $I_\varepsilon(a) = A$. Then $H_\varepsilon(a)$ is a full hereditary sub-C^*-algebra in A. By Lemma 2.12 there is a projection f in A with $(a-\varepsilon)_+ \precsim f$; and by Lemma 2.9, f is equivalent to a projection $e \in H_\varepsilon(a)$. As $(a-\varepsilon)_+ \precsim e$ and $(a-\varepsilon)_+ \perp e$ we can use Lemma 3.6 to find a projection $p \in A$ such that $\|(a-\varepsilon)_+(1-p)\| \le \varepsilon$, whence $\|a(1-p)\| \le 2\varepsilon \le 3\varepsilon$.

Suppose now that $I_\varepsilon(a) \ne A$. Let $e \in H_\varepsilon(a)$, $I = \overline{AeA}$, and $f \in A/I$ be as in Lemma 4.1, and let $\pi\colon A \to A/I$ denote the quotient mapping. Note that $\pi\big((1-e)A(1-e)\big) = \pi(A)$. It follows from Proposition 3.11 that f lifts to a projection q in $(1-e)A(1-e)$. Consider the element $x = (a-\varepsilon)_+(1-e-q) = (a-\varepsilon)_+(1-q)$, which belongs to I because $\pi(x) = 0$. Hence $x^*x \precsim e$ by pure infiniteness of A, and x^*x is clearly orthogonal to e. As both e and x^*x belong to the corner C^*-algebra $(1-q)A(1-q)$ and the relation $x^*x \precsim e$ also holds relatively to this corner, it follows from Lemma 3.6 that there is a projection r in $(1-q)A(1-q)$ such that $\|x^*x(1-r)\| \le \varepsilon^2$, whence

$$\|(a-\varepsilon)_+(1-e-q)(1-r)\| = \|x(1-r)\| \le \|x^*x(1-r)\|^{1/2} \le \varepsilon.$$

Recall that $e \perp q$ and $r \perp q$. Put $p = r + q$, and note that $(1-e-q)(1-r) = (1-e)(1-q)(1-r) = (1-e)(1-p)$. We can now deduce that

$$\begin{aligned} \|a(1-p)\| &\le \|a(1-e)(1-p)\| + \|ae(1-p)\| \\ &\le \|a(1-e-q)(1-r)\| + \|ae\| \\ &\le 2\varepsilon + \varepsilon \;=\; 3\varepsilon. \end{aligned}$$

□

Our theorem above generalizes, in the separable case, Zhang's theorem (from [22]) that all simple, purely infinite C^*-algebras are of real rank zero. The primitive ideal space of a simple C^*-algebra consists of one point (the 0-ideal) and hence trivially has a basis of compact-open sets, and any simple C^*-algebra is automatically K_0-liftable (as remarked below Definition 3.1).

Corollary 4.3 *Let A be any separable C^*-algebra.*

(i) $\mathrm{RR}(A \otimes \mathcal{O}_\infty) = 0$ *if and only if* $\mathrm{Prim}(A)$ *has a basis consisting of compact-open sets and A is K_0-liftable.*

(ii) *The following three conditions are equivalent:*

  (a) $\mathrm{RR}(A \otimes \mathcal{O}_2) = 0$,

  (b) *$A \otimes \mathcal{O}_2$ has property (IP),*

  (c) $\mathrm{Prim}(A)$ *has a basis consisting of compact-open sets.*

  If, in addition, A is purely infinite, then conditions (a)–(c) *above are equivalent to:*

  (d) *A has property (IP).*

Proof: The C^*-algebras $A \otimes \mathcal{O}_\infty$ and $A \otimes \mathcal{O}_2$ are purely infinite and separable (cf. [12] and Remark 2.6). The ideal lattices $\mathrm{Ideal}(A)$, $\mathrm{Ideal}(A \otimes \mathcal{O}_2)$, and $\mathrm{Ideal}(A \otimes \mathcal{O}_\infty)$ are isomorphic, cf. Lemma 3.4 and its proof, whence—by separability—$\mathrm{Prim}(A)$, $\mathrm{Prim}(A\otimes\mathcal{O}_\infty)$ and $\mathrm{Prim}(A\otimes\mathcal{O}_2)$ are homeomorphic. It follows from Lemma 3.4 that $A\otimes\mathcal{O}_2$ is K_0-liftable, and that $A \otimes \mathcal{O}_\infty$ is K_0-liftable if and only if A is K_0-liftable. The claims of the corollary now follow from Theorem 4.2 and Proposition 2.11. □

Extensions of separable C^*-algebras with property (IP) need not have property (IP) (not even after being tensored by the compacts), cf. [15]. But in the purely infinite case we have the following:

Corollary 4.4 *Let $0 \to I \to A \to B \to 0$ be an extension of separable C^*-algebras.*

(i) *If* $\mathrm{Prim}(I)$ *and* $\mathrm{Prim}(B)$ *have basis for their topology consisting of compact-open sets, then so does* $\mathrm{Prim}(A)$.

(ii) *If I and B are purely infinite and with property (IP), then so is A.*

Proof: (i). It suffices to show that $\mathrm{RR}(A \otimes \mathcal{O}_2) = 0$, cf. Corollary 4.3. But

$$0 \longrightarrow I \otimes \mathcal{O}_2 \longrightarrow A \otimes \mathcal{O}_2 \longrightarrow B \otimes \mathcal{O}_2 \longrightarrow 0$$

is exact, because $\mathcal{O}_2$ is exact, $\mathrm{RR}(I \otimes \mathcal{O}_2) = 0$, $\mathrm{RR}(B \otimes \mathcal{O}_2) = 0$ by Corollary 4.3, and $K_1(I \otimes \mathcal{O}_2) = 0$. It therefore follows from [6, Theorem 3.14 and Proposition 3.15] that $\mathrm{RR}(A \otimes \mathcal{O}_2)$ is zero.

(ii). It follows from (i) and Corollary 4.3 that $\mathrm{RR}(A\otimes\mathcal{O}_2) = 0$, whence A has property (IP), again by Corollary 4.3. It is shown in [12, Theorem 4.19] that extensions of purely infinite C^*-algebras again are purely infinite. □

It is shown in [3, Proposition 2.16] that $\mathrm{Prim}(A \otimes B)$ is homeomorphic to $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ when either A or B is exact. It follows in particular that $\mathrm{Prim}(A \otimes B)$ has a basis of compact-open sets if both $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ have basis of compact-open sets and if and one of A and B is exact.

The tensor product $A \otimes B$ can contain unexpected ideals if both A and B are non-exact. More specifically, it follows from a theorem of Kirchberg that if C is a simple C^*-algebra and H is an infinite-dimensional (separable) Hilbert space, then $B(H) \otimes C$ has more than the three obvious ideals (counting the two trivial ones) if and only if C is non-exact. Part (i) of the proposition below shows that $\mathrm{Prim}(A \otimes B)$ can be much larger than $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$.

Proposition 4.5 *There are separable (necessarily non-exact) C^*-algebras A and C such that $\mathrm{Prim}(A)$ consists of two points (i.e., A is an extension of two simple C^*-algebras) and $\mathrm{Prim}(C)$ consists of one point (i.e., C is simple) such that:*

(i) $\mathrm{Prim}(A \otimes C)$ *does not have a basis for its topology consisting of compact-open sets; in particular,* $\mathrm{Prim}(A \otimes C)$ *is infinite.*

(ii) *The C^*-algebras $A \otimes \mathcal{O}_2$ and $C \otimes \mathcal{O}_2$ are purely infinite and of real rank zero (and hence with property (IP)), but their tensor product $(A \otimes \mathcal{O}_2) \otimes (C \otimes \mathcal{O}_2)$ does not have property (IP) (and hence is not of real rank zero).*

Proof: Let C be the non-exact, simple, unital, separable C^*-algebra with stable rank one and real rank zero constructed by Dadarlat in [9] (see also [17, 2.1]). Let A be the (also non-exact) separable sub-C^*-algebra of $B(H)$ constructed in [17, Theorem 2.6]. Then $A \otimes C$, and hence also $A \otimes C \otimes \mathcal{O}_2$, contain more than three ideals (including the two trivial ones) (by [17, Theorem 2.6]).

It follows from [17, Proposition 2.2] (following Dadarlat's construction) that there is a UHF-algebra B which is shape equivalent to C, whence the following holds: For any C^*-algebra D, the subsets of $\mathrm{Ideal}(B \otimes D)$ and of $\mathrm{Ideal}(C \otimes D)$, consisting of all ideals that are generated by projections, are order isomorphic.

The ideal lattice of $A \otimes B \otimes \mathcal{O}_2$ is order isomorphic to the ideal lattice of A (because $B \otimes \mathcal{O}_2$ is simple and exact), so $A \otimes B \otimes \mathcal{O}_2$ has three ideals (including the two trivial ideals), and each of these three ideals is generated by its projections. It follows that $A \otimes C \otimes \mathcal{O}_2$ also has precisely three ideals that are generated by projections. Hence $A \otimes C \otimes \mathcal{O}_2$ has at least one ideal which is not generated by projections. We conclude that $A \otimes C \otimes \mathcal{O}_2$ does not have property (IP). Hence $\mathrm{Prim}(A \otimes C)$ does not have a basis of compact-open sets (by Corollary 4.3) and $(A \otimes \mathcal{O}_2) \otimes (C \otimes \mathcal{O}_2)$, which is isomorphic to $A \otimes C \otimes \mathcal{O}_2$, does not have property (IP). It follows from Corollary 4.3 that $A \otimes \mathcal{O}_2$ and $C \otimes \mathcal{O}_2$ both are of real rank zero. $\square$

Proposition 4.6 *Let A and B be C^*-algebras with property (IP). Assume that A is exact and that B is purely infinite. Then $A \otimes B$ is purely infinite and with property (IP).*

Proof: Since B is purely infinite and with property (IP), Proposition 2.14 implies that B is strongly purely infinite. But a recent result of Kirchberg says that if C and D are C^*-algebras such that one of C or D is exact and the other is strongly purely infinite, then $C \otimes D$ is strongly purely infinite (see [10]). Hence, by this result of Kirchberg it follows that $A \otimes B$ is strongly purely infinite, and hence purely infinite. Also, since A is exact and A and B have property (IP), by [17, Corollary 1.3] (based on another result of Kirchberg), it follows that $A \otimes B$ has property (IP). □

There are well-known examples of two separable nuclear C^*-algebras each of real rank zero whose minimal tensor product is a C^*-algebra not of real rank zero (see [14]). This phenomenon is eliminated when tensoring with $\mathcal{O}_2$:

Corollary 4.7 *Let A and B be separable C^*-algebras with property (IP) (or of real rank zero). Assume that A is exact. Then $A \otimes B \otimes \mathcal{O}_2$ is of real rank zero.*

Proof: It follows from Proposition 4.6 that $A \otimes B \otimes \mathcal{O}_2$ is purely infinite and with property (IP), whence this C^*-algebra is of real rank zero by Corollary 4.3. □

The two conditions (on the primitive ideal space and on K_0-liftability) in Theorem 4.2 are independent. There are purely infinite C^*-algebras that are K_0-liftable while others are not, and there are purely infinite C^*-algebras whose primitive ideal space has a basis of compact-open sets, and others where this does not hold. All four combinations exist. The C^*-algebras $C([0,1]) \otimes \mathcal{O}_\infty$ and $C([0,1]) \otimes \mathcal{O}_2$ are purely infinite with primitive ideal space homeomorphic to $[0,1]$, and this space does not have a basis of compact-open sets (i.e., is not totally disconnected); the latter C^*-algebra is K_0-liftable and the former is not (consider the surjection $C([0,1]) \otimes \mathcal{O}_\infty \to C(\{0,1\}) \otimes \mathcal{O}_\infty$). More examples are given below:

Example 4.8 (The case where the primitive ideal space is finite) Every subset of a finite T_0-space is compact (has the Heine-Borel property), so if A is a C^*-algebra for which $\mathrm{Prim}(A)$ is finite, then $\mathrm{Prim}(A)$ has a basis of compact open sets. Suppose that $\mathrm{Prim}(A)$ is finite and that A is purely infinite. Then $\mathrm{Ideal}(A)$ is a finite lattice, and there exists a decomposition series

$$0 = I_0 \lhd I_1 \lhd I_2 \lhd \cdots \lhd I_n = A,$$

where each I_j is a closed two-sided ideal in A, and where each successive quotient I_j/I_{j-1}, $j = 1, 2, \ldots, n$, is simple.

It follows from Theorem 4.2 that A is of real rank zero if and only if A is K_0-liftable (when A is separable). Actually, one can obtain this result (also in the non-separable case) from Zhang's theorem, which tells us that I_j/I_{j-1} is of real rank zero for all j, being simple and purely infinite, and from Brown and Pedersen's extension result in [6, Theorem 3.14 and Proposition 3.15], applied to the extension

$$0 \longrightarrow I_{j-1} \longrightarrow I_j \longrightarrow I_j/I_{j-1} \longrightarrow 0,$$

which yields that $\mathrm{RR}(I_j) = 0$ if (and only if) $\mathrm{RR}(I_{j-1}) = 0$ and $K_0(I_j) \to K_0(I_j/I_{j-1})$ is surjective. Hence $\mathrm{RR}(A) = 0$ if and only if $K_0(I_j) \to K_0(I_j/I_{j-1})$ is surjective for all $j = 1, 2, \dots, n$. The latter is equivalent to A being K_0-liftable (as one easily can deduce from Lemma 3.3).

In the case where $n = 2$ we have an extension $0 \to I \to A \to B \to 0$, where I and B are purely infinite C^*-algebras. Here $\mathrm{RR}(A) = 0$ if and only if the map $K_0(A) \to K_0(B)$ is surjective, or equivalently, if and only if the index map $\delta\colon K_0(B) \to K_1(I)$ is zero. Let G_0, G_1, H_0, H_1 be arbitrary countable abelian groups and let $\delta\colon G_0 \to H_1$ be any group homomorphism. Then there are stable Kirchberg algebras I and B in the UCT-class such that $K_j(B) \cong G_j$ and $K_j(I) \cong H_j$, and an essential extension $0 \to I \to A \to B \to 0$ whose index map $K_0(B) \to K_1(I)$ is conjugate to δ.

In particular, if G_0, H_1, and δ are chosen such that δ is non-zero, then A is not K_0-liftable and hence not of real rank zero; but A is K_0-liftable and of real rank zero whenever δ is zero. Evidently, both situations can occur.

Let us finally note that $\mathrm{Prim}(A)$, if finite, is Hausdorff if and only if the topology on $\mathrm{Prim}(A)$ is the discrete topology, which happens if and only if A is the direct sum of n simple purely infinite C^*-algebras. Here, K_0-liftability is automatic. Note also that $\mathrm{Prim}(A)$ is totally disconnected (meaning that all connected components are singletons) if and only if $\mathrm{Prim}(A)$ is Hausdorff.

Acknowledgments: The first named author was partially supported by NSF Grant DMS-0101060 and also by a FIPI grant from the University of Puerto Rico.

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